ON SCHRÖDINGER SYSTEMS WITH LOCAL AND NONLOCAL NONLINEARITIES - PART2

HICHEM HAJAIEJ

ABSTRACT. In this second part, we establish the existence of special solutions of the nonlinear Schrödinger system studied in the first part when the diamagnetic field is nul. We also prove some symmetry properties of these ground states solutions.

CONTENTS

1. STUDY OF GROUND STATE SOLUTIONS

1.1. **Introduction and historical remarks.** In this section, we shall study the existence and symmetry of ground states for the following $m \times m$ nonlinear Schrödinger system without magnetic field, in presence of local an nonlocal nonlinearities

(1.1)
$$\begin{cases} -\Delta\Phi_j + (\lambda - V(|x|))\Phi_j - g_j(|x|, |\Phi_1|^2, \ldots, |\Phi_m|^2)\Phi_j - \sum_{i=1}^{m} W_{ij} * h(|\Phi_i|)\frac{h'(|\Phi_j|)}{|\Phi_j|}\Phi_j = 0 \\ \text{for } 1 \leq j \leq m. \end{cases}$$

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Ipeit (Institut preparatoire aux etudes d'ingenieur de Tunis) 2, Rue Jawaher Lel Nahru - 1089 Montfleury - Tunis, Tunisie. E-mail: *hichem.hajaiej@gmail.com*.

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For every $\Phi = (\Phi_1, \dots, \Phi_m) \in \mathcal{H}^1(\mathbb{R}^N)$, we define the energy functional

$$\mathcal{E}(\Phi) = \frac{1}{2}\sum_{j=1}^{m}\int |\nabla \Phi_j|^2\, dx - \frac{1}{2}\int V(|x|)|\Phi|^2\, dx - \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) dx$$
$$- \frac{1}{2}\sum_{i,j=1}^{m}\iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|)dxdy.$$

We are interested to solve the following minimization problem

$$I_c = \inf_{\Phi \in \mathcal{S}_c} \mathcal{E}(\Phi), \qquad \mathcal{S}_c = \Big\{\Phi \in \mathcal{H}^1(\mathbb{R}^N) : \sum_{j=1}^{m}\int |\Phi_j|^2 = c\Big\}, \tag{1.2}$$

where $c > 0$ is a fixed number.

2. Main assumptions

2.1. **Assumptions on local nonlinearities.** We assume that the following conditions hold

$(V0)$ $V : \mathbb{R}^N \to \mathbb{R}^+$ satisfies

$$V(|x|) \geq V(|y|), \quad \text{for all } x, y \in \mathbb{R}^N \text{ with } |x| \leq |y| \,.$$

Moreover,

$$V(|x|) \to 0, \quad \text{as } |x| \to \infty.$$

$(G0)$ $G : (0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ is a super-modular function, namely

$$G(r, y + he_i + ke_j) + G(r, y) \geq G(r, y + he_i) + G(r, y + ke_j) \tag{2.1}$$

$$G(r_1, y + he_i) + G(r_0, y) \leq G(r_1, y) + G(r_0, y + he_i) \tag{2.2}$$

for $i \neq j$, $h, k > 0$, $y = (y_1, \dots, y_m)$ and $\{e_i\}$ is the standard basis in $\mathbb{R}^m$, $r > 0$ and $0 < r_0 < r_1$.

$(G1)$ There exists $K > 0$ such that, for all $r > 0$ and $s_1, \dots, s_m \geq 0$, we have

$$0 \leq G(r, s_1, \dots, s_m) \leq K\Big(\sum_{j=1}^{m} s_j + \sum_{j=1}^{m} s_j^{\frac{\ell_j+2}{2}}\Big), \qquad 0 < \ell_j < \frac{4}{N}.$$

$(G2)$ for all $\varepsilon > 0$, there exist $R_0 > 0$ and $S_0 > 0$ such that $G(r, s_1, \dots, s_m) \leq \varepsilon \sum_{j=1}^{m} s_j$, for all $r > R_0$ and $s_1, \dots, s_m < S_0$;

$(G3)$ For any $r > 0$, $s_1, \dots, s_m$ and $t > 1$,

$$G(r, ts_1, \dots, ts_m) \geq tG(r, s_1, \dots, s_m).$$

$(G4)$ There exist $B, \gamma, R_2, S_2 > 0$ such that

$$G(r, s_1, 0, \dots, 0) \geq B s_1^{\gamma}, \quad \text{for any } r > R_2,\ 0 \leq s_1 \leq S_2,$$

where $1 \leq \gamma < 1 + \frac{2}{N}$.

2.2. **Assumptions on the nonlocal nonlinearities.** We need the following assumptions

$(h0)$ $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, non-decreasing, $h(0) = 0$ and there exists $M > 0$ such that

$$h(s) \leq M s^{\mu} \quad \text{where} \quad 2 \leq \mu < 2 - \tfrac{1}{q} + \tfrac{2}{N};$$

$(h1)$ $h(ts) \geq t h(s)$, for all $t > 1$ and $s \geq 0$.

$(h2)$ There exist $A, S_1 > 0$ and $\beta \geq \mu$ such that $h(s) \geq A s^{\beta}$, for any $0 \leq s \leq S_1$.

$(W1)$ There exist $\Gamma, C, t_1 > 0$ such that

$$W_{11}\left(\frac{r}{t}\right) \geq C \frac{t^{\Gamma}}{r^{\Gamma}}, \quad \text{for any } r \geq 0,\ 0 \leq t \leq t_1$$

where $2N - N\beta - \Gamma + 2 > 0$.

3. Sign of the Lagrange multiplier

We have the following

Proposition 3.1. *Let $c > 0$ and assume that the minimization problem* (1.2) *admits a solution $\hat{\Phi} \in \mathcal{S}_c$ with negative energy, namely*

$$\mathcal{E}(\hat{\Phi}) = I_c < 0.$$

Assume furthermore that the function

$$N(\Phi) = \int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)dx + \frac{1}{2} \sum_{i,j=1}^{m} \iint W_{ij}(|x-y|) h(|\Phi_i(x)|) h(|\Phi_j(y)|) dx dy$$

satisfies over $\hat{\Phi}$ the condition

$$N'(\hat{\Phi}_1, \dots, \hat{\Phi}_m)(\hat{\Phi}_1, \dots, \hat{\Phi}_m) - 2N(\hat{\Phi}_1, \dots, \hat{\Phi}_m) \geq 0. \tag{3.1}$$

Let λ_c denote the Lagrange multiplier associated with $\hat{\Phi}$. Then $\lambda_c < 0$.

Proof. Of course, we have $\mathcal{E}'(\hat{\Phi}) = \lambda_c \hat{\Phi}$, so that

$$\mathcal{E}'(\hat{\Phi})(\hat{\Phi}) = \lambda_c (\hat{\Phi}, \hat{\Phi})_{\mathcal{L}^2} = \lambda_c \|\hat{\Phi}\|^2_{\mathcal{L}^2} = c\lambda_c.$$

Then, we have

$$c\lambda_c - 2I_c = \mathcal{E}'(\hat{\Phi})(\hat{\Phi}) - 2\mathcal{E}(\hat{\Phi}) = -N'(\hat{\Phi})(\hat{\Phi}) + 2N(\hat{\Phi}) = \tau,$$

namely $\lambda_c = \frac{2I_c}{c} + \frac{\tau}{c} < 0$, as $\tau \leq 0$ and $I_c < 0$ by assumption. This proves the assertion. □

Remark 3.2. Assume that the function $\mathbb{R}^m \ni s \mapsto G(r, s) \in \mathbb{R}^+$ is homogeneous of degree $\varrho \geq 1$ and $W_{ij}(x) = 0$ for all $i, j = 1, \ldots, m$ and $x \in \mathbb{R}^N$. Then condition (3.1) is satisfied. In fact, taking into account that $\nabla G(s) \cdot s = dG(s)(s) = \varrho G(s)$, it follows that

$$\begin{aligned} N'(\hat{\Phi})(\hat{\Phi}) - 2N(\hat{\Phi}) &= 2 \int \sum_{j=1}^{m} D_{s_j} G(|x|, |\hat{\Phi}_1|^2, \ldots, |\hat{\Phi}_m|^2) |\hat{\Phi}_j|^2 dx \\ &\quad - 2 \int G(|x|, |\hat{\Phi}_1|^2, \ldots, |\hat{\Phi}_m|^2) dx \\ &= 2(\varrho - 1) \int G(|x|, |\hat{\Phi}_1|^2, \ldots, |\hat{\Phi}_m|^2) dx \geq 0, \end{aligned}$$

which proves the desired claim. The homogeneity of G is often fulfilled in the applications. Think, instance, to the literature of weakly coupled nonlinear Schrödinger systems.

Remark 3.3. Assume that the function $s \mapsto h(s)$ is homogeneous of degree $\mu \geq 2$ and that $G = 0$. Then condition (3.1) is satisfied. In fact, taking into account that $h'(s)s = \mu h(s)$, by direct computation, exchanging i and j and

x with y, it follows that

$$
\begin{aligned}
N'(\hat\Phi)(\hat\Phi) - 2N(\hat\Phi) &= \frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\hat\Phi_i(x)|)h'(|\hat\Phi_j(y)|)|\hat\Phi_j(y)|dxdy \\
&+ \frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\hat\Phi_j(y)|)h'(|\hat\Phi_i(x)|)|\hat\Phi_i(x)|dxdy \\
&- \sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\hat\Phi_i(x)|)h(|\hat\Phi_j(y)|)dxdy \\
&= \sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\hat\Phi_i(y)|)h'(|\hat\Phi_j(x)|)|\hat\Phi_j(x)|dxdy \\
&- \sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\hat\Phi_i(x)|)h(|\hat\Phi_j(y)|)dxdy \\
&= (\mu-1)\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\hat\Phi_i(x)|)h(|\hat\Phi_j(y)|)dxdy \geq 0,
\end{aligned}
$$

which proves the claim. The homogeneity of h is often fulfilled in the applications. Think for instance to the literature of the *Pekar-Choquard* equation with $h(s) = |s|^\mu$, being the classical formulation in the particular case $\mu = 2$.

4. Existence and symmetry of solutions

We have the following

Proposition 4.1. *Assume conditions (V0), (G1), (h0) hold. Then, for all* $c > 0$*, problem* (1.2) *is well-posed, that is* $I_c > -\infty$*.*

Proof. Let $\Phi \in \mathcal{S}_c$. In the following, we shall denote by C a generic positive constant, possibly depending on c, that can change from line to line. From assumption (G1), we have

$$
\int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)dx \leq C + C\sum_{j=1}^m \|\Phi_j\|_{\ell_j+2}^{\ell_j+2}. \tag{4.1}
$$

From the Gagliardo-Nirenberg inequality, and since $\|\Phi_j\|_{L^2} \leq \sqrt{c}$, we have

$$
\|\Phi_j\|_{\ell_j+2}^{\ell_j+2} \leq C\|\Phi_j\|_{L^2}^{(1-\sigma_j)(\ell_j+2)}\|\nabla\Phi_j\|_{L^2}^{\sigma_j(\ell_j+2)} \leq C\|\nabla\Phi_j\|_{L^2}^{\sigma_j(\ell_j+2)}, \qquad \sigma_j = \frac{N\ell_j}{2(\ell_j+2)},
$$

for $j = 1, \dots, m$. Notice that, by assumption, we have

$$\sigma_j(\ell_j + 2) = \frac{N\ell_j}{2} < 2, \quad \text{for } j = 1, \dots, m.$$

Then, by means of Young inequality, for all $\varepsilon > 0$ there exists $K_1(\varepsilon) > 0$ such that

$$\|\Phi_j\|_{\ell_j+2}^{\ell_j+2} \le K_1(\varepsilon) + \varepsilon \|\nabla \Phi_j\|_{L^2}^2. \tag{4.2}$$

In turn, inequality (4.1) yields

$$\int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2) dx \le K_1(\varepsilon) + \varepsilon \sum_{j=1}^m \|\nabla \Phi_j\|_{L^2}^2, \tag{4.3}$$

for some positive constant $K_1(\varepsilon)$. Dealing with the nonlocal nonlinearities, from assumption (h), by the Hardy-Littlewood inequality combined with the Gagliardo-Nirenberg inequality, for any $i, j = 1, \dots, m$, since $\max\{\|W_{ij}\|_{L^q_w} : i, j = 1, \dots, m\} < \infty$, setting

$$\hat{q} = \frac{2q}{2q-1}, \qquad \gamma = \frac{N}{2}\left(\frac{\hat{q}\mu - 2}{\hat{q}\mu}\right),$$

for every $\varepsilon > 0$ there exists $K_2(\varepsilon) > 0$ such that

(4.4)

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|) h(|\Phi_i(x)|) h(|\Phi_j(y)|) dx dy \le C \sum_{i,j=1}^m \|W_{ij}\|_{L^q_w} \|\Phi_i^\mu\|_{L^{\hat{q}}} \|\Phi_j^\mu\|_{L^{\hat{q}}} \\
&\le C \sum_{i,j=1}^m \|\Phi_i\|_{L^{\hat{q}\mu}}^\mu \|\Phi_j\|_{L^{\hat{q}\mu}}^\mu \le C \sum_{i,j=1}^m \|\Phi_i\|_{L^2}^{(1-\gamma)\mu} \|\nabla \Phi_i\|_{L^2}^{\gamma\mu} \|\Phi_j\|_{L^2}^{(1-\gamma)\mu} \|\nabla \Phi_j\|_{L^2}^{\gamma\mu} \\
&\le C \sum_{i,j=1}^m \|\nabla \Phi_i\|_{L^2}^{\gamma\mu} \|\nabla \Phi_j\|_{L^2}^{\gamma\mu} \le C \sum_{i=1}^m \|\nabla \Phi_i\|_{L^2}^{2\gamma\mu} \le K_2(\varepsilon) + \varepsilon \sum_{i=1}^m \|\nabla \Phi_i\|_{L^2}^2,
\end{aligned}$$

where in the last two inequalities we used the Young inequality. In particular, the last one was possible since, by our assumptions on μ in $(h0)$, we have

$$2\gamma\mu = N\left(\frac{\hat{q}\mu - 2}{\hat{q}}\right) = N\left(\frac{q\mu - 2q + 1}{q}\right) < 2.$$

Then, fixed $\varepsilon \in (0, 1/4)$, by combining (4.3) and (4.4), by the definition of $\mathcal{E}$ and denoted by $\rho = V(0) > 0$, we have

$$\begin{aligned}
\mathcal{E}(\Phi) \geq \frac{1}{2}\sum_{j=1}^{m} \|\nabla\Phi_j\|_{L^2}^2 - \frac{\rho}{2}\sum_{j=1}^{m} \|\Phi_j\|_{L^2}^2 - \int G(|x|, |\Phi_1|^2, \ldots, |\Phi_m|^2)dx \\
- \frac{1}{2}\sum_{i,j=1}^{m} \iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|)dxdy
\end{aligned} \tag{4.5}$$

$$\geq \left(\frac{1}{2} - 2\varepsilon\right)\sum_{j=1}^{m} \|\nabla\Phi_j\|_{L^2}^2 - \frac{\rho c}{2} - K_1(\varepsilon) - K_2(\varepsilon) \geq -\frac{\rho c}{2} - K_1(\varepsilon) - K_2(\varepsilon). \tag{4.6}$$

for all $\Phi \in \mathcal{S}_c$, yielding the desired conclusion. □

The next proposition shows that, even in the limiting cases with respect to the growths of the local and nonlocal nonlinearities the minimization problem is well posed, provided that the infimum is taken over a sphere of sufficiently small radius c.

Proposition 4.2. *Assume conditions (V0), (G1), (h0) hold and that*

$$\textit{either } \ell_{j_0} = \frac{4}{N} \textit{ for some } j_0 = 1, \ldots, m \textit{ or } \mu = 2 - \frac{1}{q} + \frac{2}{N}.$$

Then $I_c > -\infty$ for every $c > 0$ sufficiently small.

Proof. Let $c > 0$ and take $\Phi \in \mathcal{S}_c$. In the following, we shall denote by C a generic positive constant which can change from line to line and which is independent of c. In fact, differently from the proof of Proposition 4.1, here we need to put c into evidence in the estimates in order to show that problem (1.2) is well posed, for all c sufficiently small. Assume that there exists $1 \leq j_0 \leq m$ such that $\ell_{j_0} = \frac{4}{N}$ (and that $\ell_j < 4/N$ for all $j \neq j_0$). Recall that $\|\Phi_{j_0}\|_{L^2} \leq \sqrt{c}$. From (G1), the Gagliardo-Nirenberg inequality

and (4.2) (holding, indeed, when $\ell_j < 4/N$), we have

$$\begin{aligned}\int G(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2)dx &\le C + C\|\Phi_{j_0}\|_{\ell_{j_0}+2}^{\ell_{j_0}+2} + C\sum_{j\neq j_0}^m \|\Phi_j\|_{\ell_j+2}^{\ell_j+2} \\ &\le C + C\|\Phi_{j_0}\|_{L^2}^{\frac{4}{N}}\|\nabla\Phi_{j_0}\|_{L^2}^2 + K_1(\varepsilon) + \varepsilon\sum_{j\neq j_0}^m \|\nabla\Phi_j\|_{L^2}^2 \\ &\le K_1(\varepsilon) + Cc^{\frac{2}{N}}\|\nabla\Phi_{j_0}\|_{L^2}^2 + \varepsilon\sum_{j\neq j_0}^m \|\nabla\Phi_j\|_{L^2}^2 \\ &\le K_1(\varepsilon) + \max\{Cc^{\frac{2}{N}}, \varepsilon\}\sum_{j=1}^m \|\nabla\Phi_j\|_{L^2}^2\end{aligned}$$

for some positive constant $K_1(\varepsilon)$ depending on ε. Concerning the nonlocal nonlinearities, we observe that, if $\mu < 2 - 1/q + 2/N$, we are in the case of the proof of Proposition 4.1 and we have inequality (4.4). If, instead, we are in the limiting case $\mu = 2 - 1/q + 2/N$, for $\hat{q} = \frac{2q}{2q-1}$ it holds

$$\gamma = \frac{1}{\mu} = \frac{Nq}{2Nq - N + 2q}.$$

In turn, by Hardy-Littlewood and Gagliardo-Nirenberg inequalities, we have

$$\begin{aligned}&\frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|)dxdy \\ &\le C\sum_{i,j=1}^m \|\Phi_i\|_{L^2}^{(1-\gamma)\mu}\|\nabla\Phi_i\|_{L^2}\|\Phi_j\|_{L^2}^{(1-\gamma)\mu}\|\nabla\Phi_j\|_{L^2} \\ &\le Cc^{(1-\gamma)\mu}\sum_{i,j=1}^m \|\nabla\Phi_i\|_{L^2}\|\nabla\Phi_j\|_{L^2} \le Cc^{(1-\gamma)\mu}\sum_{i=1}^m \|\nabla\Phi_i\|_{L^2}^2.\end{aligned}$$

In any case, by (4.4) and the above inequality, we can always write

$$\frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(|\Phi_i(x)|)h(|\Phi_j(y)|)dxdy \le \max\{Cc^{(1-\gamma)\mu}, \varepsilon\}\sum_{i=1}^m \|\nabla\Phi_i\|_{L^2}^2 + K_2(\varepsilon).$$

Then, by the definition of $\mathcal{E}$ and previous inequalities, denoted by $\rho = V(0) > 0$, we have

$$\mathcal{E}(\Phi) \ge \Big(\frac{1}{2} - \max\{Cc^{\frac{2}{N}}, \varepsilon\} - \max\{Cc^{(1-\gamma)\mu}, \varepsilon\}\Big)\sum_{j=1}^m \|\nabla\Phi_j\|_{L^2}^2 - \frac{\rho c}{2} - K_1(\varepsilon) - K_2(\varepsilon),$$

for all $\Phi \in \mathcal{S}_c$. By choosing $\varepsilon > 0$ and $c > 0$ so small that

$$\frac{1}{2} - \max\{Cc^{\frac{2}{N}}, \varepsilon\} - \max\{Cc^{(1-\gamma)\mu}, \varepsilon\} > 0$$

it holds $\mathcal{E}(\Phi) \geq -\frac{\rho c}{2} - K_1(\varepsilon) - K_2(\varepsilon)$ and the assertion follows, namely there exists $c_0 > 0$ such that the minimization problem is well posed for all $c \in (0, c_0)$. $\square$

The next proposition says that, at least under suitable assumptions, which include some classical situations, such as $h(s) = s^\mu$, $W_{ij}(x) = |x|^{-\alpha}$ and

$$G(|x|, s_1, \dots, s_m) = \frac{1}{\ell+2} \sum_{i,j=1}^{m} |s_i|^{(\ell+2)/2} + 2|s_i|^{(\ell+2)/4}|s_j|^{(\ell+2)/4},$$

the upper bounds on ℓ_j and μ are optimal for the minimization problem to be well posed.

Proposition 4.3. *Assume ($V0$) and that either there exists a function $H : \mathbb{R}^m_+ \to \mathbb{R}$, homogeneous of degree $\frac{\ell+2}{2}$ with $\ell > 4/N$, such that*

$$G(|x|, s_1, \dots, s_m) \geq H(s_1, \dots, s_m), \quad \textit{for all } (s_1, \dots, s_m) \in \mathbb{R}^m_+$$

or there exist two constants $\gamma_1, \gamma_2 > 0$ such that, for some $1 \leq i_0, j_0 \leq m$,

$$W_{i_0 j_0}(x) \geq \gamma_1 |x|^{-\alpha} \textit{ and } h(s) \geq \gamma_2 s^\mu \textit{ for all } x \in \mathbb{R}^N \textit{ and } s \in \mathbb{R}^+, \textit{ with } \mu > 2 - \frac{\alpha}{N} + \frac{2}{N}.$$

Then $I_c = -\infty$ for every $c > 0$.

Proof. We consider the case when both the situations indicated in the statement occur, the proof being similar in the other cases. Let $c > 0$ and consider a fixed function Φ_0 in $\mathcal{S}_c$. For all $t > 0$, we define the function $\Phi_t : \mathbb{R}^N \to \mathbb{R}^m$ by setting $\Phi_t^j(x) = t^{N/2}\Phi_0^j(tx)$ for all $x \in \mathbb{R}^N$ and $j = 1, \dots, m$. It follows that $\Phi_t \in \mathcal{S}_c$ for all $t > 0$, so that, by definition of

I_c, it holds for all $t > 0$ large

$$
\begin{aligned}
I_c \le \mathcal{E}(\Phi_t) &\le \frac{1}{2}\sum_{j=1}^{m}\|\nabla \Phi_t^j\|_{L^2}^2 - \int G(|x|, |\Phi_t^1|^2, \dots, |\Phi_t^m|^2)dx \\
&- \frac{1}{2}\sum_{i,j=1}^{m}\iint W_{ij}(|x-y|)h(|\Phi_t^i(x)|)h(|\Phi_t^j(y)|)dxdy \\
&\le \frac{t^2}{2}\sum_{j=1}^{m}\|\nabla \Phi_0^j\|_{L^2}^2 - t^{\frac{N\ell}{2}}\int H(|\Phi_0^1|^2, \dots, |\Phi_0^m|^2)dx \\
&- \frac{\gamma_1\gamma_2^2}{2}t^{\alpha+N\mu-2N}\iint |x-y|^{-\alpha}|\Phi_0^{i_0}(x)|^{\mu}|\Phi_0^{j_0}(y)|^{\mu}dxdy \\
&\le C_1t^2 - C_2t^{\frac{N\ell}{2}} - C_3t^{\alpha+N\mu-2N} + C_4 \\
&\le C_1t^2 - C_5t^{\min\{\frac{N\ell}{2}, \alpha+N\mu-2N\}} + C_4.
\end{aligned}
$$

By assumptions $\min\{\frac{N\ell}{2}, \alpha + N\mu - 2N\} > 2$ and the assertion follows by letting $t \to \infty$. $\square$

Proposition 4.4. *Assume conditions ($V0$), (W), ($h0$), ($G0$), ($G1$) and ($G2$) hold. Then, for every $c > 0$, problem* (1.2) *admits a minimization sequence (Φ_n) having a Schwarz symmetric weak limit Φ_0 such that $\mathcal{E}(\Phi_0) \le I_c$.*

Proof. Let $\Phi_n \in \mathcal{H}^1$ be a minimizing sequence for (1.2). Since $\|\nabla|\Phi_{n,j}|\|_{L^2} = \|\nabla\Phi_{n,j}\|_{L^2}$, we have that $\mathcal{E}(|\Phi_n|) \le \mathcal{E}(\Phi_n)$ so that $|\Phi_n|$ is a minimizing sequence too. In turn, without loss of generality, we may assume that the minimizing sequence is positive. Denoted by Φ_n^* the sequence of the Schwarz symmetrizations of Φ_n, we claim that $\mathcal{E}(\Phi_n^*) \le \mathcal{E}(|\Phi_n|)$ so that Φ_n^* is also a minimizing sequence for (1.2). In order to prove it, we take advantage of the following symmetrization inequalities. By [7], for every $j = 1, \dots, m$,

$$
\|\nabla\Phi_{n,j}^*\|_{L^2}^2 \le \|\nabla\Phi_{n,j}\|_{L^2}^2
$$
$$
\|\Phi_{n,j}^*\|_{L^2}^2 = \|\Phi_{n,j}\|_{L^2}^2.
$$

From the last equality, it follows that, if $\Phi_n \in \mathcal{S}_c$, then also $\Phi_n^* \in \mathcal{S}_c$. Moreover, in view of assumption ($V0$), we have that

$$
\int V(|x|)\Phi_{n,j}^2 \le \int V(|x|)(\Phi_{n,j}^*)^2.
$$

Furthermore, in view of the super-modularity assumption ($G0$), we have

$$
\int G(|x|, \Phi_{n,1}^2, \dots, \Phi_{n,m}^2)dx \le \int G(|x|, (\Phi_{n,1}^*)^2, \dots, (\Phi_{n,m}^*)^2)dx
$$

and, by assumptions (W) and $(h0)$, it follows

$$\iint W_{ij}(|x-y|)h(\Phi_{n,i}(x))h(\Phi_{n,j}(y))\,dxdy \leq \iint W_{ij}(|x-y|)h(\Phi^*_{n,i}(x))h(\Phi^*_{n,j}(y))\,dxdy,$$

for every any $i,j=1,\dots,m$. We shall denote by $\widetilde{\Phi}_n = \Phi_n^*$ a Schwarz symmetric minimizing sequence for (1.2). Observe that $\widetilde{\Phi}_n$ is bounded in $\mathcal{H}^1$. Indeed, if this was not the case, from the following inequality (see inequality (4.5) in Proposition 4.1), as $n\to\infty$, denoted by $\rho = V(0) > 0$,

$$I_c + o(1) = \mathcal{E}(\widetilde{\Phi}_n) \geq \left(\frac{1}{2} - 2\varepsilon\right)\sum_{j=1}^m \|\nabla\widetilde{\Phi}_{n,j}\|^2_{L^2} - \frac{\rho c}{2} - K_1(\varepsilon) - K_2(\varepsilon)$$

for $\varepsilon \in (0,\frac{1}{4})$, we would immediately get a contradiction. Hence, up to a subsequence, there exists $\Phi_0 \in \mathcal{H}^1$ such that $\widetilde{\Phi}_n$ converges to Φ_0 weakly in $\mathcal{H}^1$, locally strongly in $\mathcal{L}^s$ for $s < 2^*$ and almost everywhere in $\mathbb{R}^N$. We will prove that

$$\mathcal{E}(\Phi_0) \leq \liminf_{n\to\infty} \mathcal{E}(\widetilde{\Phi}_n). \tag{4.7}$$

For all $j = 1,\dots,m$, we know that

$$\int |\nabla\Phi_{0,j}|^2 \leq \liminf_{n\to\infty} \int |\nabla\widetilde{\Phi}_{n,j}|^2. \tag{4.8}$$

Now, let us prove that, for every $i = 1,\dots,m$,

$$\lim_{n\to\infty}\int V(|x|)\widetilde{\Phi}^2_{n,j} = \int V(|x|)\Phi^2_{0,j}, \tag{4.9}$$

$$\lim_{n\to\infty}\int G(|x|,\widetilde{\Phi}^2_{n,1},\dots,\widetilde{\Phi}^2_{n,m}) = \int G(|x|,\Phi^2_{0,1},\dots,\Phi^2_{0,m}), \tag{4.10}$$

and for all $i,j=1,\dots,m$,

$$\lim_{n\to\infty}\iint W_{ij}(|x-y|)h(\widetilde{\Phi}_{n,i}(x))h(\widetilde{\Phi}_{n,j}(y)) = \iint W_{ij}(|x-y|)h(\Phi_{0,i}(x))h(\Phi_{0,j}(y)). \tag{4.11}$$

First, we prove (4.9). Fixed $R > 0$, denote by $B(R)$ the ball of radius R centered at the origin. Since $\widetilde{\Phi}_{n,j}(x) \to \Phi_{0,j}(x)$ for a.e. $x \in B(R)$ and there exists a function $b_j \in L^2(B(R))$ such that $\widetilde{\Phi}_{n,j}(x) \leq b_j(x)$ for a.e. $x \in B(R)$, by the monotonicity assumption on V in $(V0)$, we have

$$\lim_{n\to\infty}\int_{B(R)} V(|x|)\widetilde{\Phi}^2_{n,j} = \int_{B(R)} V(|x|)|\Phi_{0,j}|^2, \tag{4.12}$$

by dominated convergence. Now, fix $\varepsilon > 0$ and $j = 1, \dots, m$. Since $V(|x|) \to 0$ as $|x| \to \infty$ by assumption $(V0)$, there exists $R(\varepsilon) > 0$ such that, for all $|x| > R(\varepsilon)$ and for every $n \in \mathbb{N}$

$$\int_{B_c(R(\varepsilon))} V(|x|)\widetilde{\Phi}^2_{n,j} \le \varepsilon \int_{B_c(R(\varepsilon))} \widetilde{\Phi}^2_{n,j} \le \varepsilon c.$$

Furthermore, in a similar fashion, we have that

$$\int_{B_c(R(\varepsilon))} V(|x|)\widetilde{\Phi}^2_{0,j}(x) \le \varepsilon c.$$

By means of (4.12), choosing $R = R(\varepsilon)$, there exists $\nu_\varepsilon \in \mathbb{N}$ such that for every $n \ge \nu_\varepsilon$

$$\left| \int_{B(R(\varepsilon))} V(|x|)\widetilde{\Phi}^2_{n,j} - \int_{B(R(\varepsilon))} V(|x|)\Phi^2_{0,j} \right| < \varepsilon.$$

Thus, by combining the above inequalities, (4.9) follows. Now, we show (4.10). Fixed $R > 0$, it holds

(4.13)
$$\lim_{n\to\infty} \int_{B(R)} G(|x|, \widetilde{\Phi}^2_{n,1}, \dots, \widetilde{\Phi}^2_{n,m}) = \int_{B(R)} G(|x|, |\Phi_{0,1}|^2, \dots, |\Phi_{0,m}|^2).$$

Indeed, $\widetilde{\Phi}_{n,j}(x) \to \Phi_{0,j}(x)$ for a.e. $x \in B(R)$, and there exist m functions $f_j \in L^{l_j+2}(B(R))$ such that $\widetilde{\Phi}_{n,j}(x) \le f_j(x)$ for a.e. $x \in B(R)$. Of course $G(|x|, \widetilde{\Phi}^2_{n,1}(x), \dots, \widetilde{\Phi}^2_{n,m}(x))$ converges pointwise to $G(|x|, |\Phi_{0,1}|^2(x), \dots, |\Phi_{0,m}|^2(x))$ in $B(R)$ and, from $(G1)$,

$$G(|x|, \widetilde{\Phi}^2_{n,1}, \dots, \widetilde{\Phi}^2_{n,m}) \le K\Big(\sum_{j=1}^m f_j^2 + \sum_{j=1}^m f_j^{l_j+2} \Big) \in L^1(B(R)),$$

Assertion (4.13) then simply follows by dominated convergence. Fixed $\varepsilon > 0$, in light of [1, Lemma A.IV] and assumption $(G2)$, there exist $R(\varepsilon) \ge R_0 > 0$ and $S_0 > 0$ such that, for all $|x| > R(\varepsilon)$, $\widetilde{\Phi}_{n,j}(x) < S_0$ for every $j = 1, \dots, m$ and for all $n \in \mathbb{N}$. Hence, by $(G2)$, we have

$$\int_{B^c(R(\varepsilon))} G(|x|, \widetilde{\Phi}^2_{n,1}, \dots, \widetilde{\Phi}^2_{n,m}) \le \varepsilon \sum_{j=1}^m \int_{B^c(R(\varepsilon))} \widetilde{\Phi}^2_{n,j}(x) \le \varepsilon c.$$

Now, observe that, since $\widetilde{\Phi}_{n,j}(x) \to \Phi_{0,j}(x)$ a.e., also $\widetilde{\Phi}_{0,j}(x) < S_0$ for all $|x| > R(\varepsilon)$. Then recalling that also $\int \widetilde{\Phi}^2_{0,j} \le c$, we obtain

$$\int_{B^c(R(\varepsilon))} G(|x|, \widetilde{\Phi}^2_{0,1}, \dots, \widetilde{\Phi}^2_{0,m}) \le \varepsilon c.$$

By means of (4.13), choosing $R = R(\varepsilon)$, there exists $\nu_\varepsilon \in \mathbb{N}$ such that, for all $n \geq \nu_\varepsilon$

$$\left| \int_{B(R(\varepsilon))} G(|x|, \widetilde{\Phi}_{n,1}^2, \dots, \widetilde{\Phi}_{n,m}^2) - \int_{B(R(\varepsilon))} G(|x|, \Phi_{0,1}^2, \dots, \Phi_{0,m}^2) \right| < \varepsilon.$$

Hence (4.10) is proved too. Finally, we come to the proof of (4.11). We know that, since $\widetilde{\Phi}_{n,j}$ is a sequence of radial functions, bounded in $H^1(\mathbb{R}^N)$, by [1, Theorem A.I'], up to a subsequence, $\widetilde{\Phi}_{n,j} \to \Phi_{0,j}$ strongly in $L^{\hat{q}\mu}(\mathbb{R}^N)$ as $n \to \infty$, where $\hat{q} = \frac{2q}{2q-1}$ and $2 < \hat{q}\mu < 2^*$. Then, there exists a function $a_j \in L^{\hat{q}\mu}(\mathbb{R}^N)$ such that, $\widetilde{\Phi}_{n,j}(x) \leq a_j(x)$ for a.e. $x \in \mathbb{R}^N$. By the continuity of h, for a.e. $x, y \in \mathbb{R}^N$ we have

$$\lim_{n\to\infty} W_{ij}(|x-y|)h(|\widetilde{\Phi}_{n,i}(x)|)h(|\widetilde{\Phi}_{n,j}(y)|) = W_{ij}(|x-y|)h(|\Phi_{0,i}(x)|)h(|\Phi_{0,j}(y)|).$$

Furthermore, since h is non-decreasing, we have for a.e. $x, y \in \mathbb{R}^N$

$$W_{ij}(|x-y|)h(|\widetilde{\Phi}_{n,i}(x)|)h(|\widetilde{\Phi}_{n,j}(y)|) \leq W_{ij}(|x-y|)h(a_i(x))h(a_j(y))$$

where the right hand side function is in $L^1(\mathbb{R}^{2N})$ by means of Hardy-Littlewood Sobolev inequality

$$\iint W_{ij}(|x-y|)h(a_i(x))h(a_j(y))dxdy \leq \|a_i\|^\mu_{L^{\hat{q}\mu}(\mathbb{R}^N)} \|W_{ij}\|_{L^q_w(\mathbb{R}^N)} \|a_j\|^\mu_{L^{\hat{q}\mu}(\mathbb{R}^N)}.$$

Then, by (4.8)-(4.11), (4.7) is proved. This yields $\mathcal{E}(\Phi_0) \leq I_c$, concluding the proof. □

Proposition 4.5. *Assume conditions ($G3$), ($h0$) and ($h1$). If $I_c < 0$, then $\mathcal{E}(\Phi_0) = I_c$ for every $c > 0$.*

Proof. In view of Proposition 4.4, we know that $\mathcal{E}(\Phi_0) \leq I_c$ and $\|\Phi_0\|^2_{\mathcal{L}^2} \leq c$. It is sufficient to prove that $\Phi_0 \in \mathcal{S}_c$. First, we observe that, by (G) and (h), $\mathcal{E}(\mathbf{0}) = 0$ then $\Phi_0 \neq \mathbf{0}$. Otherwise, by the negativity assumption on I_c, we would have

$$0 = \mathcal{E}(\Phi_0) \leq I_c < 0,$$

then a contradiction. Define $t = \frac{c^{1/2}}{\|\Phi_0\|_{\mathcal{L}^2}}$, we have that $t\Phi_0 \in \mathcal{S}_c$ and, by $\|\Phi_0\|^2_{\mathcal{L}^2} \leq c$, $t \geq 1$. So, by ($G3$), ($h1$) and Proposition 4.4, we have that

$$\mathcal{E}(t\Phi_0) = \frac{1}{2}\sum_{j=1}^m \|\nabla(t\Phi_{0,j})\|^2_{L^2} - \frac{1}{2}\sum_{j=1}^m V(x)\|t\Phi_{0,j}\|^2_{L^2} - \int G(|x|, t^2\Phi_{0,1}^2, \dots, t^2\Phi_{0,m}^2)dx$$
$$- \frac{1}{2}\sum_{i,j=1}^m \iint W_{ij}(|x-y|)h(t\Phi_{0,i}(x))h(t\Phi_j(y))dxdy \leq t^2\mathcal{E}(\Phi_0) \leq t^2 I_c.$$

Thus, $I_c \leq t^2 I_c$ and, by the negativity assumption on I_c, we have that $t \leq 1$. Hence, $t = 1$ and by the definition of t, $\|\Phi_0\|^2_{\mathcal{L}^2} = c$ thus proving the thesis. □

5. Negativity of I_c

The following results provides sufficient conditions in order to get the condition that the minimum value is negative for all values of c.

Proposition 5.1. *Assume conditions ($V0$), ($W1$) and either condition ($G4$) or condition ($h2$). Then $I_c < 0$ for all $c > 0$.*

Proof. In the following we shall assume both ($G4$) and ($h2$). It will be clear by the argument that follows that only one of these assumptions is actually sufficient to provide the desired conclusion. Given $c > 0$, we fix a positive function ϕ in $L^\infty(\mathbb{R}^N)$ such that $\|\phi\|^2_{L^2} = c$. Then, setting $\Phi = (\phi, 0, \dots, 0) \in \mathcal{H}^1$, of course we, have $\Phi \in \mathcal{S}_c$. Now, for all $0 < t < 1$, let us define $\phi_t(x) = t^{N/2}\phi(tx)$ and set $\Phi_t(x) = (\phi_t(x), 0, \dots, 0)$. Clearly, $\|\phi_t\|^2_{L^2} = c$ and $\Phi_t \in \mathcal{S}_c$, for all $0 < t < 1$. If we now evaluate the energy functional $\mathcal{E}$ at Φ_t, by a change of variable and exploiting the assumptions, for every $0 < t < \min\{t_1, \frac{1}{R_2}\}$ sufficiently small, we have that

$$0 \leq t^{N/2}\phi(x) \leq t^{N/2}\|\phi\|_{L^\infty} \leq S_1, \quad 0 \leq t^N\phi^2(x) \leq t^N\|\phi\|^2_{L^\infty} \leq S_2,$$

with S_1, S_2 and R_2 in assumptions ($G4$) and ($h2$) so that

$$\begin{aligned}
\mathcal{E}(\Phi_t) &= \frac{1}{2}\int |\nabla\phi_t(x)|^2\,dx - \frac{1}{2}\int V(|x|)\phi_t^2(x)\,dx - \int G(|x|, \phi_t^2(x), 0, \dots, 0)dx \\
&\quad - \frac{1}{2}\iint W_{11}(|x-y|)h(\phi_t(x))h(\phi_t(y))dxdy \\
&= \frac{t^2}{2}\int |\nabla\phi(x)|^2\,dx - \frac{1}{2}\int V\left(\frac{|x|}{t}\right)\phi^2(x)\,dx - t^{-N}\int G\Big(\frac{|x|}{t}, t^N\phi^2(x), 0, \dots, 0\Big)dx \\
&\quad - \frac{t^{-2N}}{2}\iint W_{11}\left(\frac{|x-y|}{t}\right)h(t^{N/2}\phi(x))h(t^{N/2}\phi(y))\,dxdy \\
&\leq \frac{t^2}{2}\int |\nabla\phi(x)|^2\,dx - t^{-N}\int_{\{|x|\geq 1\}} G\Big(\frac{|x|}{t}, t^N\phi^2(x), 0, \dots, 0\Big)dx \\
&\quad - \frac{t^{-2N}}{2}\iint W_{11}\left(\frac{|x-y|}{t}\right)h(t^{N/2}\phi(x))h(t^{N/2}\phi(y))\,dxdy \\
&\leq Dt^2 - Et^{-N}t^{N\gamma} - Ft^{-2N}t^{\Gamma}t^{N\beta},
\end{aligned}$$

where we have set

$$D := \frac{1}{2}\|\nabla\phi\|_{L^2}^2, \quad E := B\int_{\{|x|\geq 1\}}\phi^{2\gamma}dx, \quad F := A^2C\iint\frac{\phi^\beta(x)\phi^\beta(y)}{|x-y|^\gamma}dxdy.$$

In conclusion, for t small enough, we get

$$I_c \leq \mathcal{E}(\Phi_t) \leq t^2\big(D - Et^{N\gamma-N-2} - Ft^{\Gamma+N\beta-2N-2}\big),$$

where, by the assumptions of γ, β and Γ,

$$N\gamma - N - 2 < 0 \quad \text{and} \quad \Gamma + N\beta - 2N - 2 < 0.$$

By taking $t > 0$ sufficiently small, we have that $I_c \leq \mathcal{E}(\Phi_t) < 0$, proving the assertion. $\square$

Remark 5.2. Notice that, if W is a typical convolution kernel of the form $W(x) = |x|^{-\Gamma}$, it follows that W belongs to the space $L^q_w(\mathbb{R}^N)$ where $q = \frac{N}{\Gamma}$. Moreover, thinking about the important model situation $h(s) = s^\mu$, we have $\beta = \mu$. Then, we have

$$\Gamma+N\beta-2N-2 < 0 \quad \Leftrightarrow \quad \frac{N}{q}+N\mu-2N-2 < 0 \quad \Leftrightarrow \quad \mu < 2-\frac{1}{q}+\frac{2}{N},$$

which is the condition on h we are already familiar with.

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IPEIT (INSTITUT PREPARATOIRE AUX ETUDES D'INGENIEUR DE TUNIS)
2, RUE JAWAHER LEL NAHRU - 1089 MONTFLEURY - TUNIS TUNISIE
E-mail address: `hichem.hajaiej@gmail.com`